TRINITY COLLEGE

IN NOMINE DOMINI

Semester One Examination, 2020

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	99	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (99 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

A seafood processor buys batches of n prawns from their supplier, where n is a constant. In any given batch, the probability that a prawn is export quality is p, where p is a constant and the quality of an individual prawn is independent of other prawns.

The discrete random variable X is the number of export quality prawns in a batch and the mean of X is 79.2 and standard deviation of X is 6.6.

(a) State the name given to the distribution of X and determine its parameters n and p. (4 marks)

Solution
X follows a binomial distribution. $np = 79.2$ $np(1-p) = 6.6^2$ $n = 176, \quad p = \frac{9}{20} = 0.45$
Specific behaviours
✓ names binomial distribution ✓ equation for mean and variance (or sd) ✓ value of n ✓ value of p

(b) Determine the probability that more than 50% of prawns in a randomly selected batch are export quality. (2 marks)

Solution
$50\% \times 176 = 88$ $P(X \geq 89) = 0.0797$
Specific behaviours
✓ lower bound ✓ probability

Question 10 **(8 marks)**

The voltage, V volts, supplied by a battery t hours after timing began is given by

$$V = 8.95e^{-0.265t}$$

(a) Determine

(i) the initial voltage. (1 mark)

Solution
$V(0) = 8.95\text{ V}$
Specific behaviours
✓ correct value

(ii) the voltage after 3 hours. (1 mark)

Solution
$V(3) = 4.04\text{ V}$
Specific behaviours
✓ correct value

(iii) the time taken for the voltage to reach 0.03 volts. (1 mark)

Solution
$t = 21.5\text{ h}$
Specific behaviours
✓ correct value

(b) Show that $\dfrac{dV}{dt} = aV$ and state the value of the constant a. (2 marks)

Solution
$\dfrac{dV}{dt} = -0.265(8.95e^{-0.265})$ $= aV$ $a = -0.265$
Specific behaviours
✓ correct derivative   ✓ value of a

(c) Determine the rate of change of voltage 3 hours after timing began. (1 mark)

Solution
$\dot{V} = -0.265 \times 4.04 = -1.07\text{ V/h}$
Specific behaviours
✓ correct rate

(d) Determine the time at which the voltage is decreasing at 5% of its initial rate of decrease. (2 marks)

Solution
$\dot{V} \propto V \Rightarrow e^{-0.265t} = 0.05$ $t = 11.3\text{ h}$
Specific behaviours
✓ indicates suitable method   ✓ correct time

Question 11 **(8 marks)**

A small body moving in a straight line has displacement x cm from the origin at time t seconds given by

$$x = 5\cos(2t - 1) + 6.5, \qquad 0 \le t \le 3.$$

(a) Use derivatives to justify that the maximum displacement of the body occurs when $t = 0.5$. (4 marks)

Solution

$$\frac{dx}{dt} = -10\sin(2t - 1)$$

$$t = 0.5 \Rightarrow \frac{dx}{dt} = -10\sin(0) = 0$$

Hence when $t = 0.5, x$ has a stationary point.

$$\frac{d^2x}{dt^2} = -20\cos(2t - 1)$$

$$t = 0.5 \Rightarrow \frac{d^2x}{dt^2} = -20\cos(0) = -20$$

Since second derivative is negative, the stationary point is a maximum, and so the body has a maximum displacement when $t = 0.5$.

Specific behaviours

✓ first derivative
✓ indicates stationary point at required time
✓ value of second derivative at required time
✓ statement that justifies maximum

(b) Determine the time(s) when the velocity of the body is not changing. (2 marks)

Solution

$$a = \frac{d^2x}{dt^2} = -20\cos(2t - 1)$$

$$a = 0 \Rightarrow \cos(2t - 1) = 0$$

$$t = \frac{\pi}{4} + \frac{1}{2}, \frac{3\pi}{4} + \frac{1}{2} \approx 1.285, 2.856 \text{ seconds}$$

Specific behaviours

✓ indicates acceleration/second derivative must be zero
✓ states exact (or approximate) times in interval

(c) Express the acceleration of the body in terms of its displacement x. (2 marks)

Solution

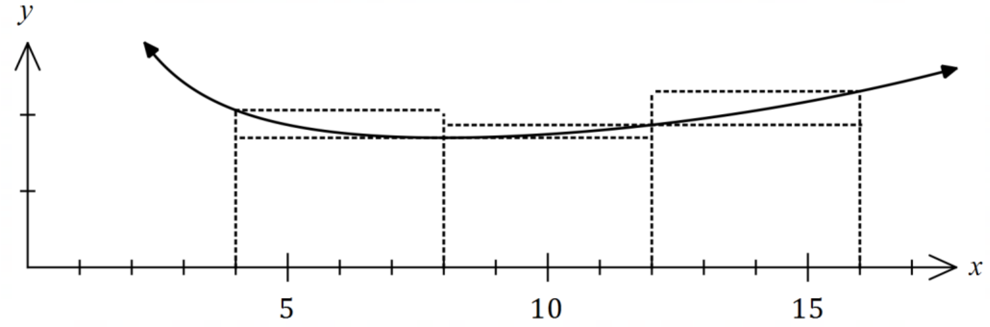
$$\begin{aligned} a &= -20\cos(2t - 1) \\ &= -4(5\cos(2t - 1)) \\ &= -4(x - 6.5) \end{aligned}$$

Specific behaviours

✓ factors out -4
✓ correct expression

Question 12 **(7 marks)**

The function f is defined as $f(x) = \dfrac{5e^{0.125x}}{x}$, $x > 0$, and the graph of $y = f(x)$ is shown below.



(a) Complete the missing values in the table below, rounding to 2 decimal places. (1 mark)

x	4	8	12	16
$f(x)$	**2.06**	1.70	1.87	**2.31**

Solution
See table
Specific behaviours
✓ both correct

(b) Use the areas of the rectangles shown on the graph to determine an under- and over-estimate for $\int_4^{16} f(x)\,dx$. (3 marks)

Solution
$U = 4(1.70 + 1.70 + 1.87) = 4 \times 5.27 = 21.08$
$O = 4(2.06 + 1.87 + 2.31) = 4 \times 6.24 = 24.96$
Specific behaviours
✓ indicates $\delta x = 4$
✓ under-estimate
✓ over-estimate

(c) Use your answers to part (b) to obtain an estimate for $\int_4^{16} f(x)\,dx$. (1 mark)

Solution
$E = (21.08 + 24.96) \div 2 \approx 23.0$
Specific behaviours
✓ correct mean

(d) State whether your estimate in part (c) is too large or too small and suggest a modification to the numerical method employed to obtain a more accurate estimate. (2 marks)

Solution
Estimate is too large ($f(x)$ is concave upwards).
Better estimate can be found using a larger number of thinner rectangles.
Specific behaviours
✓ states too big
✓ indicates modification to improve estimate

Question 13 **(8 marks)**

A bag contains four similar balls, one coloured red and three coloured green. A game consists of selecting two balls at random, one after the other and with the first replaced before the second is drawn. The random variable X is the number of red balls selected in one game.

(a) Complete the probability distribution for X below. (3 marks)

x	0	1	2
$P(X = x)$	$\mathbf{\frac{9}{16}}$	$\mathbf{\frac{6}{16}}$	$\mathbf{\frac{1}{16}}$

Solution
$P(X=0) = \left(\frac{3}{4}\right)^2 = \frac{9}{16};\ P(X=2) = \left(\frac{1}{4}\right)^2 = \frac{1}{16};\ P(X=1) = 1 - \frac{9+1}{16} = \frac{6}{16}$ $(0.5625, 0.375, 0.0625)$
Specific behaviours
✓ one correct probability ✓ probabilities have sum of 1 ✓ all correct probabilities

(b) Determine $E(X)$ and $\mathrm{Var}(X)$. (2 marks)

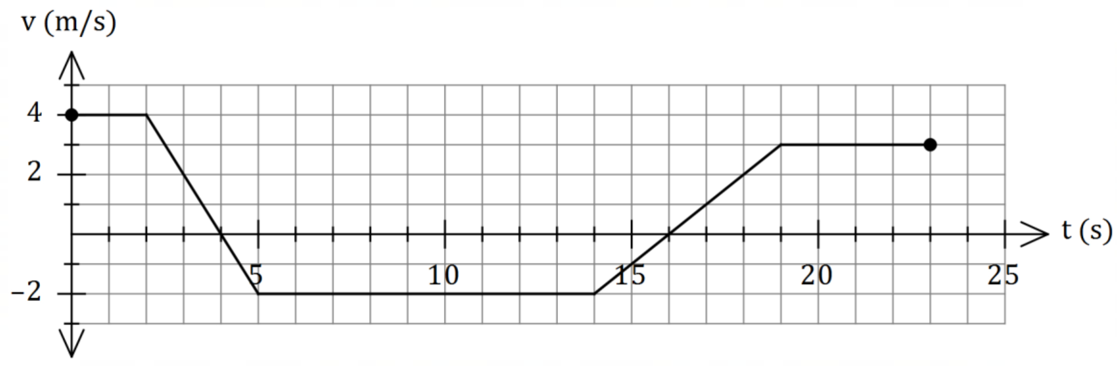
Solution
$E(X) = 0 + \frac{6}{16} + \frac{2}{16} = \frac{1}{2};\quad \mathrm{Var}(X) = \frac{3}{8} = 0.375$ *NB Using CAS,* $sd = \frac{\sqrt{6}}{4} \approx 0.6124.$
Specific behaviours
✓ expected value ✓ variance

(c) A player wins a game if the two balls selected have the same colour. Determine the probability that a player wins no more than three times when they play five games. (3 marks)

Solution
$Y \sim B\left(5, \frac{10}{16}\right)$ $P(Y \le 3) \approx 0.6185$
Specific behaviours
✓ defines distribution ✓ states probability required ✓ correct probability

Question 14 **(9 marks)**

A small body leaves point A and travels in a straight line for 23 seconds until it reaches point B. The velocity v m/s of the body is shown in the graph below for $0 \le t \le 23$ seconds.



(a) Use the graph to evaluate $\int_0^4 v\,dt$ and interpret your answer with reference to the motion of the small body. (3 marks)

Solution
$$\int_0^4 v\,dt = 2 \times 4 + \frac{1}{2} \times 2 \times 4 = 12 \text{ m}$$ The change in displacement of the body during the first 4 seconds is 12 m. OR The body has moved 12 m to the right of P during first 4 seconds.
Specific behaviours
✓ value of integral ✓ interprets as change in displacement ✓ includes specific time and distance with units in interpretation

(b) Determine an expression, in terms of t, for the displacement of the body relative to A during the interval $2 \le t \le 5$. (3 marks)

Solution
$$v = 8 - 2t \Rightarrow x = \int 8 - 2t\,dt = 8t - t^2 + c$$ $$t = 2, x = 8 \Rightarrow 8 = 8(2) - 2^2 + c \Rightarrow c = -4$$ $$x = 8t - t^2 - 4, \qquad 2 \le t \le 5$$
Specific behaviours
✓ expression for v ✓ expression for x with constant c ✓ correct expression for x

(c) Determine the time(s) at which the body was at point A for $0 < t \leq 23$. (3 marks)

Solution
$x(5) = 12 + \frac{1}{2} \times 1 \times (-2) = 11$ $11 - 2(t - 5) = 0 \Rightarrow t = 10.5$ $x(19) = -4.5$ $-4.5 + 3(t - 19) = 0 \Rightarrow t = 20.5$ Body at point A when $t = 10.5$ s and $t = 20.5$ s.
Specific behaviours
✓ indicates appropriate method using areas ✓ one correct time ✓ two correct times

Question 15 **(7 marks)**

Functions f and g are such that

$$f(2) = -1, \qquad f'(x) = 6(2x-7)^{-2}$$

$$g(-3) = -1, \qquad g'(x) = 6(2x+7)^{-2}$$

(a) Determine $f(3)$. (3 marks)

Solution
$f(3) = f(2) + \int_2^3 6(2x-7)^{-2}\,dx$ $= -1 + \left[\frac{-3}{2x-7}\right]_2^3$ $= -1 + (3-1)$ $= 1$
Specific behaviours
✓ integrates rate of change   ✓ determines change   ✓ correct value

(b) Use the increments formula to determine an approximation for $g(-2.97)$. (3 marks)

Solution
$x = -3, \qquad \delta x = 0.03$ $\delta y \approx \frac{6}{(2x+7)^2} \times \delta x$ $\approx 6 \times 0.03 \approx 0.18$ $g(-2.97) \approx -1 + 0.18 \approx -0.82$
Specific behaviours
✓ values of x and δx ✓ use of increments formula   ✓ correct approximation

(c) Briefly discuss whether using the information given about f and the increments formula would yield a reasonable approximation for $f(3)$. (1 mark)

Solution
No, approximation wouldn't - the change $\delta x = 1$ is not a small change.   *(NB Yields $f(3) \approx -\frac{1}{3}$)*
Specific behaviours
✓ states no with reason

Question 16 **(9 marks)**

When a machine is serviced, between 1 and 5 of its parts are replaced. Records indicate that 7% of machines need 1 part replaced, 8% need 5 parts replaced, 12% need 4 parts replaced, and the mean number of parts replaced per service is 2.82.

Let the random variable X be the number of parts that need replacing when a randomly selected machine is serviced.

(a) Complete the probability distribution table for X below. (4 marks)

x	1	2	3	4	5
$P(X = x)$	$\mathbf{0.07}$	$\mathbf{0.32}$	$\mathbf{0.41}$	$\mathbf{0.12}$	$\mathbf{0.08}$

Solution

Let $P(x = 2) = a, P(X = 3) = b$ then

$$0.27 + a + b = 1$$
$$0.07 + 2a + 3b + 0.48 + 0.4 = 2.82$$

Hence

$$a = 0.32, \qquad b = 0.41$$

Specific behaviours

✓ values for $x = 1, 4, 5$
✓ equation using sum of probabilities
✓ equation using expected value
✓ values for $x = 2, 3$

(b) Determine $\mathrm{Var}\,(X)$. (2 marks)

Solution

Using CAS, $\sigma = 1.00379281$

Hence $Var(X) = \sigma^2 = 1.0076$

Specific behaviours

✓ indicates sd using CAS
✓ correct variance

The cost of servicing a machine is $\$56$ plus $\$12.50$ per part replaced and the random variable Y is the cost of servicing a randomly selected machine.

(c) Determine the mean and standard deviation of Y. (3 marks)

Solution

$$Y = 56 + 12.5X$$

$$E(Y) = 56 + 12.5 \times 2.82 = \$91.25$$

$$\sigma_Y = 12.5 \times 1.00379 \approx \$12.55$$

Specific behaviours

✓ equation relating X and Y
✓ mean
✓ standard deviation (penalty no units: -1 mark)

Question 17 **(6 marks)**

Some values of the polynomial function f are shown in the table below:

x	-2	-1	0	1	2	3	4
$f(x)$	-8	0	5	6	4	1	-3

(a) Evaluate $\int_1^4 f'(x)\,dx$. (2 marks)

Solution
$\int_1^4 f'(x)\,dx = f(4) - f(1)$ $= -3 - 6$ $= -9$
Specific behaviours
✓ uses fundamental theorem   ✓ correct value

The following is also known about $f'(x)$:

Interval	$-2 \le x \le 1$	$x = 1$	$1 \le x \le 4$
$f'(x)$	$f'(x) > 0$	$f'(x) = 0$	$f'(x) < 0$

(b) Determine the area between the curve $y = f'(x)$ and the x-axis, bounded by $x = -2$ and $x = 3$. (4 marks)

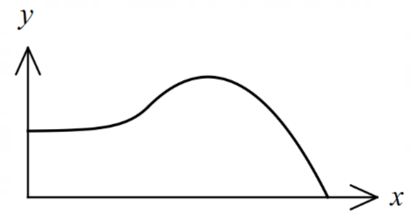
Solution
Area to left of $x = 1$ is above axis but to left is below so will need to negate/drop negative sign for that integral: $\text{Area} = \int_{-2}^{1} f'(x)\,dx - \int_1^3 f'(x)\,dx$ $= f(1) - f(-2) - [f(3) - f(1)]$ $= 2f(1) - f(-2) - f(3)$ $= 2(6) - (-8) - 1$ $= 19 \text{ sq units}$
Specific behaviours
✓ integral for $f'(x) > 0$ ✓ negated integral for $f'(x) < 0$ ✓ uses fundamental theorem   ✓ correct area

Question 18 **(6 marks)**

The edges of a swimming pool design, when viewed from above, are the x-axis, the y-axis and the curves

$y = -0.2x^2 + 3x - 6.25$ and $y = 2.75 + e^{x-5}$

where x and y are measured in metres.



(a) Determine the gradient of the curve at the point where the two curves meet. (2 marks)

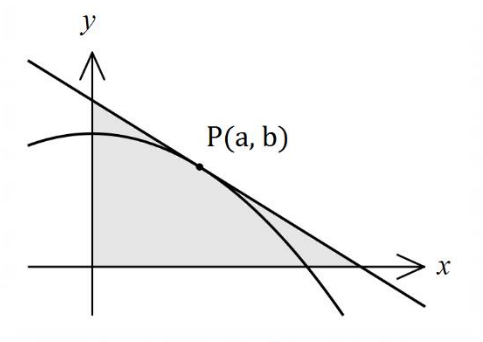
Solution
Curves intersect when $x = 5$ $y' = -0.4(5) + 3 = e^{5-5} = 1$
Specific behaviours
✓ x-coordinate of intersection   ✓ common gradient

(b) Determine the surface area of the swimming pool. (4 marks)

Solution
$A_1 = \int_0^5 2.75 + e^{x-5}\,dx = \frac{59}{4} - \frac{1}{e^5} \approx 14.743$ $A_2 = \int_5^{12.5} -0.2x^2 + 3x - 6.25\,dx = \frac{225}{8} \approx 28.125$ $A_1 + A_2 = \frac{343}{8} - \frac{1}{e^5} \approx 42.868\ \mathrm{m}^2$
Specific behaviours
✓ upper bound for parabola   ✓ area A_1 ✓ area A_2 ✓ total area, with units

Question 19 **(8 marks)**

Let $P(a,b)$ be a point in the first quadrant that lies on the curve $y=8-x^2$ and A be the area of the triangle formed by the tangent to the curve at P and the coordinate axes.



(a) Show that the equation of the tangent formed at P is given by $y=-2ax+a^2+8$. (2 marks)

Solution
Gradient at P: $\frac{dy}{dx}=-2x \Rightarrow m_P=-2a$ Equation of tangent: $y-b=-2a(x-a)$ $y-(8-a^2)=-2ax+2a^2$ $y=-2ax+a^2+8$
Specific behaviours
✓ b in terms of a and m_P ✓ equation of tangent in terms of a,x,y (any form)

(b) Hence or otherwise show that $A=\frac{(a^2+8)^2}{4a}$. (4 marks)

Solution
Axes intercepts: $y=0 \Rightarrow x=\frac{a^2+8}{2a}, \qquad x=0 \Rightarrow y=a^2+8$ Area: $A=\frac{1}{2}\left(\frac{a^2+8}{2a}\right)(a^2+8)=\frac{(a^2+8)^2}{4a}$
Specific behaviours
✓ axes intercepts ✓ indicates area of right triangle

(c) Use calculus to determine the coordinates of P that minimise A. (4 marks)

Solution
$$\frac{dA}{da}=\frac{3a^4+16a^2-64}{4a^2}$$ $$\frac{dA}{da}=0\Rightarrow a=\frac{2\sqrt{6}}{3}\approx 1.633$$ $$\frac{d^2A}{da^2}=\left.\frac{3a^4+64}{2a^3}\right\rvert_{a=\frac{2\sqrt{6}}{3}}=4\sqrt{6}\Rightarrow \text{Minimum}$$ $$b=8-a^2=\frac{16}{3}$$ Hence $P\left(\frac{2\sqrt{6}}{3},\frac{16}{3}\right)\approx P(1.633, 5.333)$
Specific behaviours
✓ first derivative ✓ solves for a ✓ indicates check for minimum (graph, sign or second derivative test) ✓ correct coordinates, exact or at least 2 dp

Question 20 **(9 marks)**

A farmer introduced a species of trout into his dam. The number of trout, P, t years after they were introduced is modelled by the equation $P = 80e^{0.2t}$ where $t \geq 0$.

(a) How many trout were initially introduced to the dam? (1 mark)

Solution
$P(0) = 80e^{0.2(0)}$ $P(0) = 80$
Specific behaviours
✓ correct value

(b) Determine the number of years taken for the population of trout to first exceed 1000. (2 marks)

Solution
$1000 = 80e^{0.2t}$ $t = 12.63$
Specific behaviours
✓ Solves for t ✓ correct value of t

(c) Calculate $\frac{dP}{dt}$ when $t = 5$ and explain the value you have calculated with reference to the context of the question. (3 marks)

Solution
$\frac{dP}{dt} = 16e^{0.2t}$ When $t = 5: \frac{dP}{dt} = 43.49$ We have calculated the instantaneous rate of change of the trout population at 5 years.
Specific behaviours
✓ calculates derivative ✓ calculates derivative @ t= 5 ✓ mentions instantaneous rate of change of population

(d) The farmer also introduced yabbies into his dam at the same time as the trout. The farmer introduced 200 yabbies and it is known the yabby population had a rate of change given by $\frac{dP}{dt} = 0.07t$. After how many years would the farmer expect the population of the trout and yabbies to be equal. (3marks)

Solution
Population equation for Yabbies: $P = 200e^{0.07t}$ Solve: $200e^{0.07t} = 80e^{0.2t}$ $t = 7.04$
Specific behaviours
✓ writes equation for Yabby population ✓ Solves for populations to be equal ✓ correct value t = 7.04

Question 21 **(8 marks)**

When a byte of data is sent through a network in binary form (a sequence of bits - 0's and 1's), there is a chance of bit errors that corrupt the byte, i.e. a 0 becomes a 1 and vice versa.

Suppose a byte consists of a sequence of 8 bits and for a particular network, the chance of a bit error is 0.300%.

(a) Determine the probability that a byte is transmitted without corruption, rounding your answer to 5 decimal places. (3 marks)

Solution
$X \sim B(8, 0.003)$ $P(X = 0) = 0.97625$
Specific behaviours
✓ indicates binomial distribution ✓ indicates probability to calculate ✓ correct probability, to 5 dp

(b) Determine the probability that during the transmission of 32 bytes, at least one of the bytes becomes corrupted. (2 marks)

Solution
$Y \sim B(32, 0.02375)$ $P(Y \geq 1) = 0.5366$
Specific behaviours
✓ indicates correct method ✓ correct probability

A Hamming code converts a byte of 8 bits into a byte of 12 bits for transmission, with the advantage that if just one bit error occurs during transmission, it can be detected and corrected.

(c) Determine the probability that during the transmission of 32 bytes using Hamming codes, at least one of the bytes becomes permanently corrupted. (3 marks)

Solution
$H \sim B(12, 0.003)$ $P(H \geq 2) = 0.00058$ $M \sim B(32, 0.00058) \Rightarrow P(M \geq 1) = 0.0185$
Specific behaviours
✓ states distribution of failures of a 12 bit byte ✓ probability that single Hamming code byte corrupted ✓ correct probability

End of questions

Supplementary page

Question number: __________